Intuitive Mathematics: Theoretical and Educational Implications

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What kinds of intuitions do people have for solving problems in a formal logic system? Studies on intuitive physics have shown that people hold a set of naive beliefs (Chi & Slotta, 1993; diSessa, 1982,1993; McCloskey, Caramazza, & Green, 1980). For example, McCloskey, Caramazza, and Green (1980) found that when people were asked to draw the path of a moving object shot through a curved tube, they believed that the object would move along a curved (instead of a straight) path even in the absence of external forces. Such an Aristotelian conceptualization of motion, although mistaken, may be based in part on forming an analogy to real-life examples, such as the Earth's circular movement around the sun (one does not "see" the forces that sustain such a movement).

Does there exist a similar body of knowledge that we can refer to as "intuitive mathematics?" That is, can we identify a set of naive beliefs that are applied to solving abstract mathematics problems? If so, how do these intuitions hinder or facilitate problem solving? The answers to these questions have implications for both psychology and education. By examining the nature of intuitive mathematics we could help (a) improve our understanding of people's formal- and informal- reasoning skills and (b) create more effective instructional materials.

The focus of much research, to date, has been on the development of early mathematical cognition. A prime example comes from Rochel Gelman and colleagues' (e.g., Gelman, 1979, 1990; Gelman & Meck, 1983; Starkey, Spelke, & Gelman, 1990) work on implicit counting principles that enable pre-school children to understand and to perform addition. In contrast, there has been much less emphasis on intuitive understanding that develops as a result of learning mathematical procedures in later years. The largest lament on the part of the education community has been that an emphasis on learning procedures can lead to rote execution of problem-solving steps, resulting in lack of correct intuition for these procedures. For example, students often learn how to execute a procedure, such as multicolumn subtraction, without understanding its underlying teleology (VanLehn, 1990). This finding has led to a proliferation of educational programs that emphasize the conceptual over the procedural (e.g., NCTM, 1989).

In this chapter, we will examine the nature and origin of what we term as *symbolic intuition*, or the intuitive understanding of mathematical symbols that develops as a result of experience with formal and abstract school-based procedures. Before we define more formally what we mean by intuition, in general, and symbolic intuition, in particular, we would like the reader to try and solve the following five problems, taken from a typical post-arithmetic mathematics curriculum:

(1) How many lines pass through any given two points in 2-dimensional space?
(2) A fair coin is flipped 10 times. The first 9 flips all come up heads. What is the probability that the next, 10th toss, will come up heads?
(3) Which set has more members, the set of all rational numbers (numbers that can be expressed as one integer over another) or the set of all irrational numbers (non-repeating decimals)?
(4) What is 8% of 142?
(5) Solve for x: "3x + 7 = 19"

Now, try and introspect about your experiences. To what extent did you have intuitions about solving these problems and what were they like? Most high-school students and adults have correct intuition about the first problem (only one line can pass through two points in 2-dimensional space), incorrect intuitions about the second and third problems (erroneously assuming that there is a greater probability that the tenth toss will come up heads, and that all infinities are alike, respectively) and no intuitions about the fourth and fifth problems because computing percentages and solving polynomials can usually be done by employing procedures without understanding of the concepts involved. We will return to discussing the correct, incorrect, and lack of intuitions for solving these specific problems in more detail later on.

In this chapter we seek to extend the work on the mathematical cognition of pre-school and elementary-school children to examining the symbolic intuitions that secondary-school students may have about problems such as the above. Do secondary school students have qualitatively different mathematical intuitions, and if so, how are these intuitions developed? As students move beyond the study of elementary arithmetic, what is the role of intuitions in their mathematics learning? In the remainder of this chapter we will discuss the nature and origin of correct and faulty intuitions in secondary school, in some detail, by identifying possible school-based and cognitive causes. We will then provide some general prescriptions towards enhancing correct symbolic intuition.

First, however, we would like to present a more formal definition of intuition. The term *intuition* is used in a variety of ways by scholars in different disciplines (see Wild, 1936, for a list of 31 definitions of intuition). Instead of listing each one, we divide intuitions according to two main views, the classical and the inferential, based on their philosophical origins . We will then integrate these two viewpoints into our own conception of symbolic intuition.

What is Mathematical Intuition?

The question of what exactly intuition is, in general, is relevant to a variety of domains, including philosophy, mathematics, psychology, and education (Westcott, 1968). Philosophers, such as Bergson and Spinoza, have contrasted intuition with reason and logic, a view that can be found in some modern conceptualizations of mathematical intuition to be discussed below. Mathematicians have traditionally regarded intuition as a way of understanding proofs and conceptualizing problems (Hadamard, 1954). Psychologists have examined the role of intuitive thinking in a variety of domains including clinical diagnosis, creativity, decision making, reasoning, and problem solving. A growing body of research in cognitive psychology has been devoted to studying the process of insight, which is defined as a sudden understanding of something, an “Aha!” experience, after a period of trying to solve a problem unsuccessfully (e.g., Davidson, 1995; Gick & Lockhart, 1995; Seifert, Meyer, Davidson, Palatano, & Yaniv, 1995). This literature views intuition as a phenomenon which primarily occurs through implicit and non-analyzable processes. The psychological study of mathematical intuition has been mainly conducted in the area of statistical reasoning (see, for example, Tversky & Kahneman, 1974). This work shows that people are susceptible to a variety of biases, such as ignoring base-rate information in making probabilistic judgments. Finally, educators have been concerned with the question of how intuition affects the school-learning process. There has been recent focus in education on uncovering students' pre-existing knowledge in order to make connections between school-taught, formal knowledge, and students' informal intuitions (Mack, 1990; Resnick, 1986). A review of existing literature in the above areas led us to identify two primary views of mathematical intuition. We call them the *classical-intuitionist* and the *inferential-intuitionist* views, based on their philosophical origins.

The classical intuitionist view. The main idea underlying the classical-intuitionist view is that mathematical intuition is dissociated from formal reasoning. That is, students represent a mathematics problem in such a way that the answer becomes self evident immediately, without the need for justification or formal analysis. This view can be traced to a philosophical movement termed "classical intuitionism," wherein philosophers such as Spinoza and Bergson argued that reason plays no role in intuition (Westcott, 1968; Wild, 1938). Classical intuitionists viewed intuition as "a special contact with prime reality, producing a sense of ultimate unity, true beauty, perfect certainty, and blessedness" (Westcott, 1968, p.22). According to this viewpoint, intuition is antithetical to reason. The knowledge gained through intuition cannot be verified, supported, or even understood intellectually. Intuitive knowledge is not practical or applicable. It is considered to be a priori and independent of prior knowledge.

Some more modern conceptualizations in psychology and education embrace similar views on intuition. For example, Resnick (1986) views mathematical intuitions as cognitive primitives that can function without formal mathematical analysis. Similarly, Dixon and Moore (1996) define intuitive understanding of a problem as a representation that is distinct from the representation of the formal solution procedure for solving the problem. These intuitive representation, they argue, can be uncovered by using estimation tasks (also see Reed, 1984). Dreyfus and Eisenberg (1982) define intuitions as mental representations of facts that appear to be self evident. They operationalize mathematical intuition as students' ability to solve problems despite the absence of formal instruction on the topic. Finally, Fischbein and colleagues (Fischbein, Tirosh, & Melamed, 1981) define *intuitive acceptance* as the act of accepting a certain solution or interpretation directly without explicit or detailed justification.

Where do such intuitions come from? A primary conjecture is that intuition that is unschooled and untutored is innate. An example of such an approach comes from Rochel Gelman and colleagues (Gelman, 1979, 1990; Gelman & Meck, 1983; Starkey, Spelke, & Gelman) who have delineated a set of what they believe are innate counting principles (this work will be elaborated on in a subsequent section on primary intuitions). Other researchers, such as Karen Wynn (1995), have argued that infants have the ability to mentally represent numbers. Furthermore, Wynn's studies show that infants tend to gaze longer at numerically incorrect versus correct outcomes of simple arithmetic calculations. Wynn argues that this finding demonstrates that infants can perform simple computations with positive integers that form the foundation of later numerical competence. The view that children possess intuitive or naive theories has been suggested in other domains as well, such as biology (Carey, 1995; Keil, 1981; Springer & Keil, 1989). The main relevance of the work on intuitive theories to the classical intuitionist view is that it proposes that intuitive theories of numerosity, simple arithmetic, and other domains, are pre-existing, and that they are not learned, developed, or acquired by induction from experience (Wynn, 1992).

The inferential intuitionist view. The inferential-intuitionist view departs radically from the classical-intuitionist view. The main idea underlying the inferential-intuitionist approach is that intuition is not a special mechanism but a form of reasoning guided by people's interactions with the environment. This view can be traced to the writings of philosophers such as Ewing and Bunge (Westcott, 1968) who treated intuition as the product of prior experience and reason. In particular, Ewing felt that intuition was no more than a justified belief, whose immediacy of apprehension (the Aha!) is only an illusion resulting from a series of rapid inferences unavailable to consciousness. Conceived in this way, intuitions could be subject to error, depending on experience. Similarly, Bunge felt that intuitions are hypotheses that people test by performing probabilistic judgments.

An example of such a view in the education literature comes from Fischbein (1973). Fischbein argues that feelings of immediacy, coherence, and confidence about a mathematical solution may be the result of a mini-theory or model which supports inferences based on implicit knowledge. In this framework, the processes which give rise to intuitions operate tacitly and without awareness, but they are suggested to be the same processes that support more explicit mathematical reasoning. Fischbein contends that through a process of training and familiarization, individuals can come to develop new intuitions. Thus, this perspective implies that intuitions can be learned, acquired, and developed.

Primary and Secondary Intuitions

The classical intuitionist view may be most useful for examining *primary* intuitions (Fischbein, 1973) that include the kind of informal everyday knowledge that pre-school children use for performing simple arithmetic such as counting and addition. The inferential intuitionist view, in contrast, may provide a more useful framework for examining *secondary* intuitions that are built up over long periods of formal training (Fischbein, 1973). This distinction between primary and secondary intuitions is important, because it has the potential to re-frame the types of questions one can ask about the development of children's mathematical intuitions.

Traditionally, research on children's mathematical intuitions have focused on articulating the principled knowledge that children have prior to formal schooling. Gelman and colleagues contend that principled knowledge appears very early in life before acquisition of language and assimilation of culture (e.g., Starkey, Spelke, & Gelman, 1990). For example, Rochel Gelman and colleagues (Gelman, 1979, 1990; Gelman & Meck, 1983; Starkey, Spelke, & Gelman) propose that preschoolers have implicit counting principles before they are able to verbalize or state these principles explicitly. They offer five counting principles: (1) The one-to-one principle: Every item in a display should be tagged with one and only one unique tag; (2) The stable order principle: The tags must be ordered in the same sequence across trials; (3) The cardinal principle: The last tag used in a count sequence is the symbol for the number of items in the set; (4) The abstraction principle: Any kinds of objects can be collected together for purposes of a count; and (5) The order-irrelevance principle: Objects in a set may be tagged in any sequence as long as the other counting principles are not violated.

In order to show that children have implicit knowledge of this sort, Gelman and colleagues have used ingenious experimental manipulations. For example, Gelman and Meck (1983) used an "error-detection" paradigm. In this paradigm, pre-school children are asked to help teach a puppet to count by indicating whether a particular count sequence was correct. To examine knowledge of the one-to-one principle, for instance, the puppet either counts correctly, commits an error (counts an item twice or skipped it altogether), or commits a pseudo-error (counts correctly but by using an unusual sequence, such as skipping back and forth between items). Nearly all of the children tested detected almost all of the puppet's errors. Most of the children offered some comment as to why the puppet was in error in each case. Although none of the children stated the one-to-one principle explicitly, their explanations indicated that they had acquired an understanding of the principle. Furthermore, children treated the pseudo-errors as peculiar but not as erroneous. This work suggests that preschoolers do not only possess general all-purpose abilities to sense and learn, but that they have capabilities to create and manipulate domain-specific representations.

In the early school years, children appear to rely on schemas for solving arithmetic word problems successfully. Schemas are mental structures that organize information from the environment in a useful way (Bartlett, 1932; Piaget, 1965; Schank & Abelson, 1977). For instance, Greeno (1980) and Riley, Greeno, and Heller (1983) suggest that children develop three types of schemas for solving arithmetic word problems. They include *change* (Joe had some marbles. Then Tom gave him 5 marbles. Now Joe has 8 marbles. How many marbles did Joe have at the beginning?), *combination* (Joe has 3 marbles. Tom has 5 marbles. How many marbles do they have together?), and *comparison* (Joe has 3 marbles. Tom has 5 marbles more than Joe. How many marbles does Tom have?). At the basis of these schemas lies the "part-whole" schema (Riley et al., 1983). This schema allows children to distinguish the whole from its parts in order to successfully solve change, combination, and comparison problems.

Children appear to have intuitive theories that enable them to understand the world and to make predictions beyond the domain of mathematics. For example, Carey (1995) has demonstrated that even young children have a theory of living beings. A common underlying belief to this theory is the fact that living things can move on their own and are composed of internal biological matter. For instance, children often decide that a toy monkey is more similar to a human being than to a worm, but when they are told that people have spleens, children often decide that the worm had a spleen but the toy monkey does not. The phenomenon where young children use naive causal theories for performing judgments that transcend surface-structural similarities has also been shown by Susan Gelman and her colleagues (Gelman, 1988; Gelman & Markman, 1987).

Altogether, the above studies demonstrate that children have principled knowledge that allows them to reason correctly and efficiently. Yet, despite their mastery of certain fundamental mathematical ideas, children continue to have great difficulty with school mathematics. Thus, the driving question in much of the current research on mathematical cognition is, "Why should strong and reliable intuitions of the kind that have been documented in young children fail to reliably sustain school mathematics learning?" (Resnick, 1986, p.161). One common answer to this question is that the school mathematics overemphasizes rote manipulation of symbols. This overly procedural focus may discourage children from using their intuitions on school-learning tasks (Hiebert & Lefevre, 1986; Resnick, 1986; Resnick, 1989).

However, if one accepts the idea that secondary intuitions exist, that it is possible to learn or develop intuitions, then it becomes plausible to imagine that mathematical concepts which are initially counter-intuitive (or concepts for which we have no intuitions) may eventually become intuitive via correct reasoning. This idea constitutes a considerable departure from the ways in which mathematical intuition has been traditionally studied. Rather than focusing on how students' primary intuitions can be applied to school-taught procedures one can instead seek to establish how new intuitions can be developed.

Reconsider the 5 problems originally posed at the beginning of this paper. The initial problem of how many lines pass through any given two points in 2-dimensional space refers to the often intuitive fact that exactly one line can be drawn through any two points. Problem solvers who correctly answer this question are most probably utilizing their primary intuition. In contrast, consider the other four problems. The second problem presents a scenario where a fair coin is flipped 10 times and the first 9 flips all come up heads. The question regards the probability that the next, 10th toss, will come up heads. The answer, .50, is not intuitive. Indeed, Kahneman & Tversky (1972) showed that people tend to judge sequences that have the same probabilities, such as HTTHTH and HHHTTT as being more or less "random." The answer to the third problem is even less intuitive. It asks whether the set of rational numbers or the set of irrational numbers have more members. People's reasoning usually leads to the mistaken conclusion that both sets are infinite and therefore have an equal number of members. Mathematics, however, suggests that the infinity of the set of irrational numbers is larger than the infinity of the set of rational numbers (the proof lies beyond the scope of this chapter, but the interested reader can consult Hofstadter, 1979 for an excellent discussion of Cantor's diagonal method) Whereas many people find the mathematically-correct solutions to the second and third problems to be counter-intuitive, they may not have intuitions about the solutions to the fourth and fifth problems, but merely possess recipes for solving these problems rotely (What is 8% of 142? and Solve for x: "$3x + 7 = 19$").

The traditional approach to developing primary mathematical intuition would involve refining the instruction that accompanies problems two through five, in order to enable students to connect new material with their pre-existing intuitions. An alternative view would be to foster the development of new, secondary intuitions. For example, Fischbein (1975) suggests that a student can develop the intuition that the probability of heads on the 10th flip is 0.5 by a process of experimentation and reflection (actually tossing a coin many times and observing the outcomes). The idea is that secondary intuitions can be distilled from engaging in active exploration and experimentation.

One reason why students' primary intuitions may not support school mathematics learning is that the nature of the mathematics learned in middle and secondary school is very different from that learned in elementary school. Students' initial exposure to post-arithmetic mathematics is often abstract and symbolic. In order to perform successfully, students are required to demonstrate facility with mathematical symbols, both procedurally and conceptually (Arcavi, 1994; Fey, 1990), and to develop a new kind of symbolic intuition.

What is the nature of symbolic intuition and how can it be fostered? There are two possible approaches to examining this question. The first is to explore people's correct intuitions. The second is to investigate people's incorrect intuitions or misconceptions. Most of the work in this field has concentrated on the latter. The reason may be that misconceptions are particularly valuable for uncovering the ways in which people understand and represent problems internally. Piaget (1965) has argued that by uncovering the origin of errors, one can learn as much, or even more, about people's mental representations, than by examining correct performance alone. We take the same approach in the following discussion. We believe that by examining the nature of errors we may be able to shed some light on mathematical intuition. At the conclusion of the paper, we return to the question of how correct symbolic intuitions may be developed as well.

Mathematical misconceptions in the school years and beyond

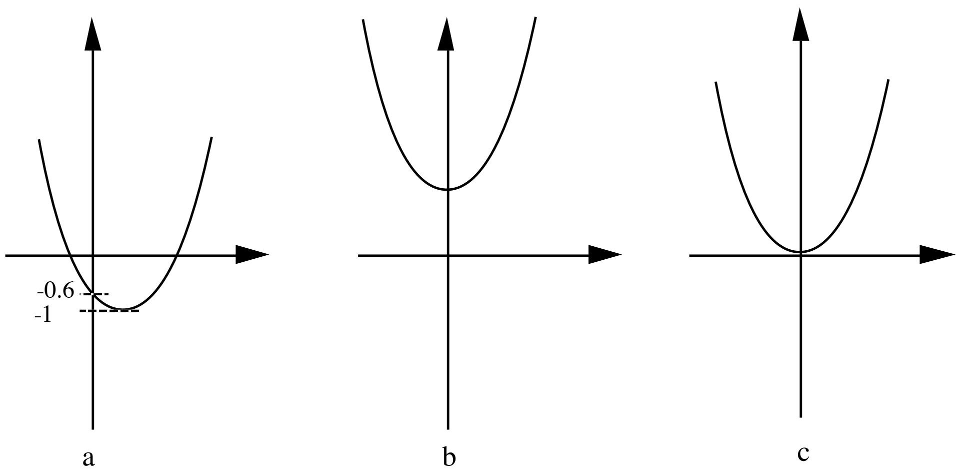
Problem-solving errors are intriguing. They are often rule based and internally consistent resulting in solutions termed *rational errors* (Ben-Zeev, 1995, 1996, in press). Rational errors are *misconceptions* that result from an active construction of knowledge, in contrast to *slips* , or careless errors that are produced from inadvertent actions in the execution of a procedure, such as omissions, permutations, and intrusions (Norman, 1981). For example, in the process of learning multicolumn subtraction, children often commit the following error:

$$\begin{array}{r} 23 \\ -\underline{\ \ 7} \\ 24 \end{array}$$

On the surface, this error appears random. A closer look, however, reveals the underlying rule: “subtract the smaller from the larger digit” (VanLehn, 1990). Specifically, the child subtracts the smaller digit (3) from the larger digit (7), irrespective of the digits’ position in the column (top or bottom). The rational basis for this error may stem from the student’s previous experience with single-digit subtraction, where the student was always taught to subtract smaller from larger numbers. The above error seems to result from applying a correct rule in an incorrect context.

Studies have demonstrated that students posses a variety of mathematical misconceptions, especially during the school years and adulthood (for a detailed review, see Ben-Zeev, 1996). Some of these misconceptions have been linked directly to “flawed” mathematical intuitions. Following are examples of several common misconceptions.

Geometrical problem-solving. Dugdale (1993) describes a geometry error that occurred in a high school class, where students were asked to match polynomials to functions (i.e., they were shown a graph and asked to choose its corresponding equation from a set of given equations). The error consisted of confusing the y-intercept of a parabola with its vertex (i.e., the “visual” center of the graph). For example, students who commit this error would decide that the y-intercept in the parabola marked “a” below is “-1,” instead of “-.6.” Dugdale explains the confusion between the y-intercept and the vertex by pointing out that in previous examples students were given, the y-intercept had always coincided with the vertex of the parabola (see the parabolas c and b below). The students had thus invented a functional invariance between the two features.



Exponential growth. Mullet and Cheminat (1995) found that high-school students tend to underestimate the rapidity of growth in exponential expressions. They provided students with examples of exponential expressions, of the form a^x; where values of *a* were 5, 7, and 9, and values of *x* were 2, 3, 4, and 5. Students were explicitly told that the expressions 5^2 and 9^5 were the smallest and largest expressions respectively. Students were then shown a sequence of exponential expressions, one at a time, and were asked to rate the magnitude of each expression on an equal-interval scale where the small and large extremes corresponded to the expressions 5^2 and 9^5, respectively. Results showed that almost all of the students exhibited difficulties in accurately estimating the value of a given exponential expression. In particular, students tended to believe that a linear increase in the exponent of an exponential expression (e.g., from 4^2 to 4^3 to 4^4) resulted in a linear increase in the entire expression (rather than an exponential increase). Similarly, students believed that a linear increase in the base of an exponential expression (e.g., from 4^3 to 5^3 to 6^3) resulted in a linear increase to the entire expression.

Weighted averages. Reed (1984) explored college students' intuitions about the concept of weighted averages, as represented in the estimation of solutions to standard distance-rate-time, work, and mixture problems. Reed found that students had great difficulty in using the weighted-average method to correctly estimate the solutions to all types of problems. Specifically, he found that students combined two numbers by calculating an unweighted average even when that average provided an unreasonable answer. For example, students were told that a pipe could fill a tank in 10 hours and another pipe could fill the same tank in 2 hours. When asked how long it would take to fill the tank if both pipes were used at the same time, about one-third of the students' response was 6 hours (the unweighted average of 2 and 10). Reed's conclusion was that many students lack an intuitive understanding of the weighted average concept.

Calculus. Several researchers have examined students' understanding of the concept of "limit" in calculus (Davis & Vinner, 1986; Dreyfus, 1990). They found that a common misconception was erroneously to assert that the terms in an infinite sequence get closer and closer to the limit but never reach it, such that for all n, $a_n \neq L$ (where a_n is the n*th* term of the sequence and L is the limit value).

Lack of intuition or (inadvertently) misleading teaching practices: An analysis of student's common misconceptions

Why do the correct intuitions that Resnick, Gelman, and others have found in young children fail to support mathematics learning in the school years? The answer to this question may lie in the interaction between the student's cognitive processing, on the one hand, and the type of instruction the student receives, on the other. Central to this argument is the idea that errors often result from a systematic and logically-consistent attempt to solve a new and unfamiliar mathematics problem (Ashlock, 1976; Brown & VanLehn, 1980; Buswell, 1926; Cox, 1975; Lankford, 1972; VanLehn, 1983). An error often makes sense to the student who created it and agrees with the student's intuitions. In this section we will focus on how teaching by using worked-out examples (Anderson, 1993; Ben-Zeev, 1995, 1996; Holland, Holyoak, Nisbett, & Thagard, 1986; Zhu & Simon, 1988; VanLehn, 1986, 1990) and schemas (Davis, 1982; Hinsley, Hayes, & Simon, 1977; Mayer, 1982, 1985; Riley, Greeno, & Heller, 1983; Ross, 1984) can lead to the formation of correct and faulty intuitions, including the misconceptions described above.

Induction from worked-out examples. Teachers use examples as concrete tools for illustrating concepts and procedures. The underlying assumption is that by following the steps in a worked-out example, students would be able to induce or generalize the correct concept or procedure for the given skill, especially when the example is specific rather than general (Sweller, & Cooper, 1985; VanLehn, 1990), and when students are encouraged to generate explanations during the learning process (Chi, Bassok, Lewis, Reimann, & Glaser, 1989; Chi, & VanLehn, 1991; VanLehn, Jones, & Chi, 1992). For instance, in teaching the multi-column subtraction algorithm, teachers find it easier to illustrate concepts such as "borrowing" by using worked-out examples. It would be quite cumbersome to state the borrowing actions verbally and in the abstract. Similarly, when students are first introduced to the representation of fractions, teachers often provide concrete examples such as: "a 1/3 is one piece of a three-piece pie."

In providing students with concrete examples, teachers are actually catering to students' own preferences. In fact, when students are given a choice between using worked-out examples versus written instructions or explanations, students tend overwhelmingly to choose the former (Anderson, Farrell, & Saurers, 1984; LeFevre & Dixon, 1986; Pirolli, & Anderson, 1985). This kind of learning, where teachers' instruction and students' learning styles match, has been termed "felicitous" (VanLehn, 1990).

When worked-out examples lead to the development of faulty intuitions. Consider a common error that students produce in the process of adding fractions:

$$\frac{1}{3} + \frac{1}{2} = \frac{2}{5}$$

This error involves adding the numerators and denominators of the fractions directly. Silver (1986) suggests that it stems from instruction that illustrates fractions as parts of a pie, as mentioned in the above section. Specifically, he claims that students may reason that because a "1/3" is one part of a three-piece pie, and a "1/2" is one part of a two-piece pie, then altogether they make 2 pieces out of total of 5 pieces, or "2/5."

This finding generalizes to adult students, as well. Consider the previous examples regarding misconceptions of exponential growth and the concept of a "limit" in calculus. Mullet and Cheminat (1995) showed that students tend to erroneously interpret linear growth in exponential expressions. Students' misconceptions may result from overgeneralization from the more familiar linear expressions. Similarly, the misconception that the terms in an infinite sequence get closer and closer to the limit but never reach it, may be based on frequently-encountered examples of infinite sequences, namely, monotonically increasing or decreasing infinite sequences such as ".1, .01, .001, .0001, ...".

The idea that students systematically overgeneralize solutions from familiar problems has received empirical support as well. Ben-Zeev (1995) instructed Yale undergraduates on performing addition in a new number system called NewAbacus. She found that when students encountered new problems on the NewAbacus addition test, they produced systematic algorithmic variations on the examples they received during the learning phase.

Schemas. In the section on mathematical intuition in the early years, we have presented Greeno and colleagues' work (Greeno, 1980; Riley, Greeno, and Heller, 1983) on the use of change, combination, and comparison schemas for solving arithmetic word problems. In teaching more advanced algebraic word problems, students learn to develop schemas as well. Hinsley, Hayes, and Simon (1979) showed that college and high school students could categorize mathematics problems into different types by using the very first words of the problem. For instance, problems that began with "A river steamer..." were categorized quickly as being part of the "river current" category. In essence, students were retrieving a schema for solving the problem by paying attention to particular salient features in the problem.

Additional evidence for schema use comes from Mayer (1982). Mayer presented students with a variety of word problems. They were either very commonly encountered problems in algebraic textbooks or were of a less common type. Mayer asked students to first read and later to recall a set of these problems. He found that when students tried to recall the less common problems, they often changed these problems' forms into the more common versions. The common problems were associated with well-formed schemas, and may have therefore formed the basis for the recall of less familiar problems.

When schemas contribute to the development of faulty intuitions. A set of studies conducted by Kurt Reusser (reported by Schoenfeld, 1991) shows the negative effects of schema use as early as the first grade. Reusser provided first- and second-graders with the following problem: "There are 26 sheep and 10 goats on a ship. How old is the captain?" The majority of students were content to respond that the captain was 36 years old. In a similar vein, Reusser asked fourth- and fifth-graders to solve the following problem: "There are 125 sheep and 5 dogs in a flock. How old is the shepherd?" This time students performed more elaborate calculations to get to a "reasonable" solution. For instance, several students attempted solving the problem by calculating "125 + 5 = 130," and "125 - 5 = 120," first. They realized, however, that these results were "too big." The students then had the "insight" of performing "125/5 = 25," and concluded that the shepherd was 25 years old.

Other instances of overusing schemas can be seen in more advanced mathematical domains. A particularly striking example comes from Paige and Simon (1966). Paige and Simon gave college students problems that were logically impossible, such as the following:

> The number of quarters a man has is seven times the number of dimes he has. The value of the dimes exceeds the value of the quarters by two dollars and fifty cents. How many has he of each coin?

This problem is logically impossible because if the number of quarters exceeds the number of dimes, then the value of the dimes cannot exceed that of the quarters. The majority of college students that were tested, however, were quite content to set up the formal equations ($Q = 7D$ and $.10D = 2.5 + .25Q$) for "solving" the problem. This kind of performance results from applying a rote schema for identifying variables and expressing the relationship between them in a formal way, without paying attention to the actual meaning of the problem.

Operator schemata and deceptive correlations. A group of schemata that involves the detection of correlations between a problem's features and the operator or algorithm that is required for solving the problem has been termed "operator-schemata" (Lewis & Anderson, 1985). The main idea underlying these schemata is that students learn explicitly or implicitly to associate a cue in a problem with the strategy for solving the problem. A particularly compelling example of this phenomenon comes from an elementary-school mathematics teacher (Schoenfeld, 1991) who taught students to explicitly search for a "cue" word in arithmetic word problems, and then to associate that cue with a particular solution strategy. Specifically, the teacher instructed the students to associate the word "left" with performing subtraction, on problems similar to the following:

Tom has 5 apples
Jerry takes away 3
How many apples are left?

However, when the same children were given the word "left" in nonsensical word problems with a similar surface structure (e.g., containing sentences such as "Tom sits to the left of Jerry"), students proceeded to subtract the given quantities in the problem, signifying that what was a well-intended strategy on the part of the teacher fostered faulty learning.

This example provides anecdotal evidence, from a real classroom environment, for what may be the effects of deceptive or spurious correlations on problem-solving performance (Ben-Zeev, 1996, 1998). These effects occur when a student perceives an association between an irrelevant feature in a problem and the strategy that is used for solving that problem (e.g., the association between the word "left" and the subtraction operator). When the student detects the irrelevant feature (the word "left") in a new problem that requires a different solution algorithm, the student may, nevertheless, proceed to carry out the correlated solution strategy erroneously.

The confusion between the y-intercept of a parabola with its vertex, presented previously, may result from such a process of detecting and using deceptive correlations. Dugdale (1993) suggested that this confusion may occur from the fact that in previous examples students were given, parabolas were symmetrical about the y-axis, resulting in a situation where the y-intercept and the vertex of the parabola lie on the same point. Students may have encoded the correlation between the y-intercept and the vertex erroneously (e.g., "when I am asked to find the value of the y-intercept, then I look for the lowest or highest point in the parabola").

There have been some empirical demonstrations of the effects of deceptive correlations on problem solving. For example, Ross (1984) taught college students elementary probability principles (e.g., permutation) by providing them with worked-out examples. Each example had a particular content (e.g., involving dice). When participants were tested on the probability principles, they tended to associate the particular problem content with the specific principle with which it had appeared in the worked-out example. When the same content appeared in a problem requiring a different probability principle, participants were "reminded" of the original principle with which the content was associated and proceeded to apply it erroneously.

More recently, Ben-Zeev (1998) demonstrated that deceptive or spurious correlations can affect the performance of even experienced problem solvers. Participants who received high scores on their Math SATs (700 or above) were instructed on how to solve problems that are frequently encountered on the Math SATs, called quantitative comparisons, by using two different algorithms, *multiply one side by n/n*, and *multiply both sides by n* (demonstrated in the Table below).

For half the participants in the study, *multiply one side by n/n* was correlated with a logarithm and *multiply both sides by n* was correlated with a radical. For the other half, the feature-algorithm correlations were flipped: *multiply one side by n/n* was correlated with a radical and *multiply both sides by n* was correlated with a logarithm. During the testing phase, in one particular experiment, participants were given an implicit memory task where they were presented with a sequence of problem-algorithm pairs on the computer screen for a very short duration (700 msec). Participants were then asked to rate the extent to which they would have "liked" or preferred solving the given problem by using the given algorithm, on a 1-7 scale. Even though most participants reported that they did not have enough time to see the stimuli on the screen, and they felt like they were guessing, results on the implicit memory task showed that participants produced higher ratings in response to algorithms that were correlated with a problem feature during learning than when the algorithm was not.

This finding shows that even on an implicit level, participants exhibited an intuitive preference for using the correlated algorithm, even when there was no conceptual reason for doing so. The reason that students may come to rely on correlational structure may be that in most cases feature-algorithm correlations are predictive cues that lead students to the correct results (e.g., explicitly associate the word "left" in a word problem with subtraction).

Algorithm 1: multiply one column by n/n

$x > 0$

Column A: $\frac{x + 4}{log 3}$

Column B: $\frac{2x + 6}{2 log 3}$

Determine whether the quantity in Column A is smaller than, larger than, or equal to the quantity in Column B, or whether the relationship cannot be determined.

Strategy: Multiply Column A by 2/2. This gives us $\frac{2x + 8}{2 log 3}$ in Column A.
By comparing the denominators we find that because $2x + 8$ is larger than $2x + 6$, then Column A is larger.

Algorithm 2: multiply both columns by n

$x > 0$

Column A: $\frac{x + 2}{\sqrt{2}}$

Column B: $\frac{2x + 4}{2\sqrt{2}}$

Determine whether the quantity in Column A is smaller than, larger than, or equal to the quantity in Column B, or whether the relationship cannot be determined.

Strategy: Multiply both columns by $\sqrt{2}$. This action cancels the $\sqrt{2}$ in both columns, and leaves us with $x + 2$ in Column A and $\frac{2x + 4}{2}$ or $x + 4$ in Column B.
Because $x + 4$ is larger than $x + 2$, then Column B is larger.

The review of the misconceptions literature above suggests that the development of incorrect intuition may be related, in part, to instructional variables such as teaching by using worked-out examples and schemas. The ontogenesis of correct symbolic intuition, on the other hand, is largely an unexplored issue.

Correct symbolic intuition

Examining the development of symbolic intuition is a significant departure from the ways in which mathematical intuition has been previously studied. Earlier research on intuition was concerned with primary intuitions and sought to determine why pre-existing intuitions failed to support school mathematics. The modal recommendation which emerged from studies on primary intuition was for teachers, schools, and curriculum developers to change instructional practices so as to make better connections with students' intuitive beliefs. The curriculum materials and new pedagogy that emerged from this program of research have made significant progress toward improving the state of mathematics learning and teaching in elementary schools and, to some extent, middle schools (e.g., NCTM, 1989).

Through this chapter, we have attempted to extend the discussion of mathematical intuition into the high school years, in particular with respect to exploring the new symbolic intuitions that students develop or fail to develop in the study of post-arithmetic mathematics. Given the significant differences in the nature of the mathematics that is learned in elementary school and high school, we believe that researchers should consider the possibility that intuitions about elementary and more advanced mathematics may be qualitatively different. A review of the current research on mathematical intuition has indicated that this issue is largely uncharted. However, we have identified two different perspectives that can inform the way in which our exploration of correct secondary intuition should proceed. Each of these perspectives may give some clarity to the study of secondary intuitions but also raises difficult questions.

A focus on deep understanding. One way to explore the development of students' secondary intuitions is to follow the trail of those who study primary intuitions. This viewpoint suggests that students develop or fail to develop secondary intuitions because of poor instructional methods and curricula. Teaching and learning should be reconceptualized, according to this view, so as to connect with and build upon students' primary intuitions.

Given the extensive use of symbols and procedures in secondary mathematics learning, the development of secondary intuition would necessarily be related to students' conceptual and procedural knowledge. Much has been written about the relationship between these two types of knowledge (e.g., Byrnes, 1992; Byrnes & Wasik, 1991; Hiebert, 1984, 1986). The main finding is that too often symbolic procedures are learned by rote and suffer from an impoverished conceptual knowledge base. The main idea is that if symbols and procedures could be imbued with and linked to conceptual knowledge, then students would be more likely to develop a deep understanding of symbolic procedures. It seems logical to assume that such deep understanding of symbolic procedures would be instrumental in the development of symbolic intuition.

This view shows promise in informing and guiding research into the development of secondary intuition. However, it also raises two difficult questions. First, the vast majority of research on conceptual and procedural knowledge has studied elementary school mathematics learning. For the most part, the conceptual knowledge underlying the learning of arithmetic procedures (adding, subtracting, multiplying, dividing) can be clearly delineated. For example, the borrowing procedure for subtraction is based on the idea of place value (Baroody, 1985), and the conceptual knowledge of fraction procedures involves understanding the part/whole relationship (Leinhardt, 1988; Mack, 1990). (Also, see Lampert, 1986, for a discussion of conceptual knowledge for multidigit multiplication). However, it is more difficult to identify or specify the conceptual knowledge that underlies algebraic procedures. For example, what does it mean to have deep conceptual understanding of the procedure which is used to solve the equation "$3x+7 = 19$?" Most mathematics educators and researchers feel that they recognize deep algebraic understanding when they see it, but they find it difficult to articulate exactly what that deep understanding just is or how to design curricula to foster its development. If the development of symbolic intuition is critically dependent on the establishment of links between conceptual and procedural knowledge of secondary school mathematics, more effort needs to be devoted to explicitly defining what is meant by these terms, particularly with respect to algebra.

Second, the focus on deep understanding fails to consider how incomplete or incorrect primary intuitions may be implicated in the development of secondary intuition. As mentioned earlier, children have been shown to have some well-developed primary intuitions upon entering school. Elementary school mathematics instruction has sought to connect and strengthen these primary intuitions. To what extent is this approach generalizable to secondary school mathematics? Is the construction of secondary intuition critically dependent on the existence of strong primary intuitions? In other words, can students develop intuitions about algebra when they have not developed intuitions about (for example) adding fractions? If instruction should seek to connect to and build upon existing intuitions, how should secondary school teachers and researchers proceed when primary intuitions are incomplete or incorrect? The relationship between existing primary intuitions and the development of symbolic intuitions has not yet been adequately examined.

A focus on doing. An alternative way in which to examine the development of secondary mathematical intuitions is more "traditionalist." According to this viewpoint, students need to initially approach their learning of symbolic mathematics with the idea that mathematical meaning may be independent of intuition. It is through the doing of (and subsequent reflection upon the doing of) symbolic procedures that students may come to develop deep understanding (and thus intuitions about) mathematical procedures (Baroody & Ginsburg, 1986).

The idea that correct learning may occur by first mastering procedures is consistent with a rich literature within the field of cognitive psychology on skill acquisition and learning by doing (e.g., Anzai & Simon, 1979). At times, this literature has been mischaracterized as advocating rote learning by proposing so-called "drill and kill" approaches to mathematics instruction. A more careful reading of research in this area indicates that not only does procedural practice play a vital role in the development of acquiring skills but it may also be instrumental in promoting conceptual understanding. Anzai and Simon (1979) found that when individuals are able to successfully execute a procedural task, even in a crude, rote, or inefficient method, they are often able to use their correct solution in order to develop more efficient and thoughtful methods of problem solving. More recently, Simon and Zhu (1988) showed that students who were given examples of the factorization of polynomials (e.g., $X^2 + 5X + 6 = (X + 2)(X + 3)$), and were then asked to solve problems on their own (e.g., $X^2 + 9X + 18 = (\ \ )(\ \ )$), were able to generalize the underlying rules of factorization correctly (i.e., $X^2 + aX + b = (X + c)(X + d)$, where $c \cdot d = b$ and $c + d = a$).

Similarly, Resnick (1986, p.191) reports the following anecdote: "The best high school math students I have talked with also have said that they are quite willing to suspend their need for "sense" for a while while new rules are introduced, because they have found that after a period of just manipulating symbols in accord with the rules, the rules come to make sense to them." The question of what it means to understand a symbolic algebraic procedure, therefore, is non trivial. It may be that suspending the need for sense in the execution of a procedure could have a beneficial role in the development of subsequent understanding and intuition. In the next section, we examine more closely the implications of intuitive mathematics for classroom learning and teaching.

Implications for the Teaching and Learning of Mathematics

How can teachers build on and foster mathematical intuition in the classroom? In an attempt to answer this questions, it is important to consider the different roles that primary and secondary intuitions play in elementary- and secondary-school mathematics. The educational questions of interest with regard to primary intuitions is whether and how should instruction be redesigned so as to connect school mathematics with students' pre-existing intuitions. As mentioned previously, many educators feel that instruction does not connect to or build upon pre-existing understanding to the extent that it could.

In elementary-school mathematics, making connections with students' primary intuitions can happen in at least two ways. First, students should be encouraged to generate and discover mathematical procedures on their own instead of only being taught the standard or most efficient problem-solving strategies by a teacher directly. This kind of a discovery approach can be used in the learning of formal mathematical algorithms, such as procedures for adding fractions and multiplying multi-digit numbers. It can also be used in less algorithmic problem-solving contexts, such as the invention of procedures to determine the relative magnitude of rational numbers or the parity (odd/even) of integers. Second, school instruction can be redesigned to foster connections to pre-existing knowledge through the use of manipulatives. Manipulatives (e.g., Geoboards, Cuisenaire rods, and base-10 cubes) allow students to contextualize or make concrete the abstract concepts that numbers represent and to form abstract mathematical relationships based on familiar and more intuitive objects and relationships.

In the case of secondary intuitions, however, these two educational strategies (i.e., discovery learning and the use of manipulatives) seem more difficult to put into practice. The act of self-generating mathematical procedures and principles may not be always possible when one is working with complex symbolic expressions or equations. Also, despite some availability of manipulatives in teaching beginning algebra concepts (e.g., algebra tiles), it is uncertain how useful or practical these manipulatives are for learning more advanced algebraic or symbolic relationships.

Thus, at first glance, the task of how to redesign instruction so as to foster the development of secondary intuitions appears to be difficult. A second look, however, may give some hope for middle and high school teachers. When students are encouraged to generate their own procedures for solving problems, what kind of reasoning do they engage in? During discovery learning, students cyclically go through the processes of experimentation followed by reflection (Fischbein, 1975). The value of the discovery approach lies not in the act of discovery itself, but rather in the thinking process that students engage in during discovery. Therefore, middle and secondary school teachers need not worry as much about whether students arrive at the right answer the first time they engage in discovery learning; rather, teachers should try to incorporate instruction which emulates the processes which underlie discovery: experimentation and reflection.

What is meant by discovery is not always apparent, even to well intending instructors or textbook authors, however. Some high-school discovery-oriented textbooks ask students to ascertain mathematical relationships that are obvious to them already. In the second author's 10^{th}-grade classroom, this situation occurred when students were solving a problem, from a Geometry text, which asked them to cut out triangles, measure each of the three angles, and find their sum. Most students already knew the answer to this exercise but many completed the activity anyway. Some students found a sum of 181 degrees or 179 degrees and yet filled in the blank on the worksheet with 180 degrees. By the end of this activity, perhaps a few more students had learned the fact that the sum of the angles in any triangle is 180 degrees. It is doubtful that this mathematical fact was any more intuitive to students as a result of their "discovery." As this situation suggests, it is important that discovery is not divorced from experimentation and reflection. Discovery learning requires interest, motivation, and intrinsic reward.

The use of manipulatives may also enable students to go through a related cyclic process of experimentation (manipulation of concrete objects), attaining a sense of the underlying principles, and articulating the problem-solving process (Mason, 1996). Manipulation provides students with a means for getting a sense of patterns, relationships, and quantities. As students start to get a sense of the material, they begin to try to bring these patterns and relationships to articulation. The act of articulation -- speaking and/or writing -- changes the way students see ideas; patterns and relationships which previously existed only in the concrete world of the manipulatives can become generalized or more abstract. This shift in the form of the ideas, from concrete to generalizable, allows what was previously abstract to become manipulable, repeating the cycle. Clearly, it is not the use of manipulatives in and of itself that can lead to the development of students' intuitions. Manipulatives can help students see the general in the specific and the specific in the general (Mason, 1996). It is this process of iterative generalization which has the most potential to encourage the development of secondary intuitions.

This iterative generalization process can be found in the Curriculum and Evaluation Standards for School Mathematics (NCTM, 1989). For example, one problem involves determining the amount of money in a person's bank account at the end of 10 years, given an $100 initial deposit and 6% interest rate, compounded annually. Students often initially attempt to manipulate the patterns and relationships in this problem with a calculator; by engaging in repeated calculations involving multiplication and addition, students can get a sense of the problem situation and articulate a solution. Further generalization can be encouraged by 'what if' questions (e.g., Brown & Walter, 1993), such as: What if the initial deposit is changed to $500? By using and manipulating a variable for the principal, students can begin to get a sense of the relationship between the initial deposit and the ending balance. This process can be repeated in a way that encourages students to continue generalizing and thus to develop secondary intuitions about the patterns and relationships in this particular problem.

In sum, our recommendations for incorporating primary and secondary intuitions into the classroom are to incorporate both discovery-oriented learning and the use of manipulatives, in the ways that we have described these two instructional strategies above. These teaching strategies can help connect new formal materials to student's pre-school knowledge as well as to foster new intuitions about repeating structures and patterns. Intuition can and should be an integral part of the learning process.

Conclusion

In this chapter we have attempted to explore the issue of secondary school students' mathematical intuitions. We have argued that the study of such intuitions may require a different approach than ones which have been reported in the well-established literature on students' intuitions in elementary school. In sum, we make the following four main points.

First, we adopt the philosophical stance that intuitions can be learned. We argue that if one accepts the idea that secondary intuitions exist, one must also acknowledge that reason does play a role in intuition (the inferential-intuitionist viewpoint). Students enter formal schooling with a wide variety of mathematical abilities and intuitions. As schooling progresses, students not only construct mathematical knowledge and strengthen existing intuitions but also build new secondary intuitions.

Second, the mathematics education and psychological communities have examined extensively the relationship between students' informal and primary mathematical intuitions and their subsequent learning in elementary school. The constructivist perspective, which argues that instructional methods should focus on allowing students to strengthen existing intuitions by making connections between informal and formal mathematical understanding, has yielded promising results (Resnick, 1986, 1992). However, it is unclear how this approach relates to the development of higher-level mathematical intuition.

Third, there has been a great deal of research devoted to the development of misconceptions. This research has shown us that systematic errors may be based on an internally-consistent logic that is overgeneralized from worked-out examples and schemas. The logic underlying errors may lead to the development of incorrect intuitions. This finding highlights the importance of instructional variables, such as teaching by using worked-out examples and by schemas, in the process of forming (mis)understandings.

Finally, we believe that the relationship between instructional variables and the development of secondary intuition is quite complex. Students fail to develop reliable, sustainable secondary mathematical intuitions for different and interrelated reasons. Among these reasons is the presence of incomplete and/or incorrect primary intuitions, dissonance between current instructional methods and students' intuitive ideas about mathematics, and the ill-defined relationship between algebraic procedural skills and deep conceptual understanding.

Mathematical intuition may be a rubric for at least two different kinds of intuitions: primary and secondary. We have already gained some advances in understanding the nature of primary mathematical intuition. The future challenge is to identify the nature of secondary or post-arithmetic mathematics intuition. Such a quest will shed some light on intuitive processes, in general, and will have important applications to education.

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