Richness of the Mathematics

Part II

National Center for Teacher Effectiveness

Center for Education Policy Research
HARVARD UNIVERSITY

Mathematical Practices Codes

- Multiple procedures or solution methods
- Developing mathematical generalizations
- Mathematical language

Multiple Procedures or Solution Methods

- *Definition*: This code is intended to capture
  - Multiple solution *methods* for a single problem (including shortcuts)
    - E.g., student A solves it one way, student B uses a different solution method
  - Multiple procedures for a given problem type
    - E.g., class compares 3/5 and 4/7 by finding a common denominator and then compares 3/5 and 6/11 by finding a common numerator

Multiple Procedures or Solution Methods

- *Distinguish from*:
  - Simply multiple representations or multiple *answers;*
  - Methods must be *distinct*
    - *Not distinct*: For comparing 3/5 and 7/10, finding common denominators of 10ths and 50ths
    - *Distinct*: Finding common numerators vs. finding common denominators
- *Note*:
  - Both methods must be correct
  - Record multiple methods when more than one method is mentioned, even if only one of them is enacted.
  - If multiple methods span several segments, only give credit in the segment(s) where the second method is discussed or enacted.

Once there are Multiple Procedures or Solution Methods: Mid or High?

- **High (3)**
  - The special features below should occur at some length
    - Explicit extended comparison of multiple procedures/solution methods for efficiency, appropriateness, ease of use, or other advantages and disadvantages OR
    - Explicit discussion of features of a problem that cue the selection of a particular procedure over another OR
    - Explicit discussion of connections between multiple procedures/solution methods (e.g., how one is like or unlike the other)
- **Mid (2)**
  - Multiple solutions present, but not linked or compared OR
  - Feature the properties under "high" only briefly
- **Low (1)**
  - No evidence of multiple methods; or they are present but incorrect

Developing Mathematical Generalizations

- *Definition*: Class examines instances or examples, *then* makes a general statement about a mathematical pattern/procedure, develops a definition, or derives a mathematical property.
- *Examples*:
  - Patterns: Drawing parabolas $y = x^2$, $y = 2x^2$, $y = 4x^2$ and then making a generalization about the shape as the coefficient changes
  - Definitions: Considering different examples and non-examples of a shape, then proposing *a definition* for this shape
  - Properties: Considering several examples of additions and subtractions (e.g., 3 + 5, 5 + 3, 9 - 4, 4 - 9), then concluding that while addition is commutative, subtraction is not

Developing Mathematical Generalizations

- *Distinguish from*: **Stating** mathematical generalizations. The generalization must be *developed*.
  - *NOT developed*: Teacher begins class by stating that addition is commutative, subtraction is not, then shows examples
  - *Developed*: Class considers several examples, then reaches a conclusion from those examples
    - Can be either *teacher* or *student* who reaches the conclusion

Now that it is a ***developed*** generalization....

- **High(3)**
  - Generalization contains the mathematical essence of the work, and is complete and clear
    - E.g., after finding the answers to 2+4, 6+10, and 12+8, a third grade class concludes that

      "The *sum* of *any* two even numbers is also an even number because
      - 2+4 means 1+2 *pairs*, which is 3 *pairs*
      - 6+10 means 3+5 *pairs*, which is 8 *pairs*
      - 12+8 means 6+4 *pairs*, which is 10 *pairs*
      - *So, in all cases, you get a certain number of pairs. And because you have pairs, the sum is an even number."*
    - Often will include mathematical explanation for the generalization, carefully chosen examples (e.g., 0, rational numbers, integers)

Developing Mathematical Generalizations

- **Mid (2)**
  - Generalization is accurate, but not complete, clear, or detailed
    - E.g., after figuring out the answers to 2+4, 6+10, and 12+8, the class concludes that "two even numbers *give us* an even number"
- **Low (1)**
  - No developed generalization OR
  - Incorrectly developed generalization OR
  - Non-mathematical generalizations
    - E.g., "Drawing a picture helps to solve a word problem."

Developing Mathematical Generalizations

- *Notes*:
  - Requires at least two examples (either explicitly worked or referred to) from which a generalization emerges
  - Record generalizations for only the segments in which generalization emerges/becomes explicit
  - Do NOT record a mid or high when teachers *state* generalizations without first developing them from examples

Mathematical Language

- *Definition*: This code is intended to capture how fluently the teacher (and students!) use mathematical language and whether the teacher supports students' use of mathematical language.
- *Examples*:
  - Talk that is dense in correct mathematical terms
  - Instruction that correctly "translates" between mathematical terms and everyday terms for mathematical phenomena
  - Teachers' efforts to encourage students to use mathematical terms

Mathematical Language

- *Distinguish from*:
  - Instruction in which teacher uses an average number of mathematical terms in pro forma way
  - Does not use mathematical terms, uses only a few, or does not use terms correctly
- Do not use this code to record students' sloppy use of language
  - Student talk can count toward a 'high' but otherwise disregard

Mathematical Language

- **High (3)**
  - Teacher uses mathematical language *fluently, densely* and in ways that help students use language. Also assign a rating of high when students use an unusually large number of mathematical terms.
  - Fluency and high density is one way to get a high
  - May also assign a rating of high for “special features” including:
    - Being explicit about or emphasizing terminology
    - “Translating” or reminding students of the meaning of terms (“discrete data, or data that can only take certain values…”)
    - Pressing students for accurate use of terms (either explicitly or via re-voicing student utterances in more precise terms)
    - Encouraging student use of mathematical terms

Mathematical Language

- **Mid (2)**
  - Reasonable amount of mathematical language in the segment, but used in pro forma way
  - This is the default rating when the teacher is using mathematical language adequately but not outstandingly
    - Teacher uses mathematical language as a vehicle for conveying content, but has few or none of the special features listed under “high”
    - The segment includes special features listed under “high” but also includes some linguistic sloppiness

Mathematical Language

- **Low (1)**
  - Teacher does not demonstrate fluency in mathematical language
    - Teacher frequently uses colloquial terms at times when more technical terms would be appropriate (e.g., “alligator mouth” for less than, “top” and “bottom” for numerator and denominator) OR
    - Teacher talk *characterized* by sloppy/inaccurate use of mathematical terms OR
    - Little mathematical language is used in the segment

Notes on Mathematical Language

- Language (richness) vs. imprecision in language (errors)
  - Richness language captures strong/fluent use of mathematical language; Imprecision captures sloppy/imprecise use of mathematical language
  - They are not simply opposites
    - Can assign a rating of “mid” in Richness even with some linguistic sloppiness and imprecise language
    - Can assign a rating of “high” in Richness with an isolated error, as long as segment meets criteria

More General Notes on Richness

- Do *not* score substantially incorrect elements as rich:
  - E.g., Incorrect or inappropriate solution method or generalizations
- Evidence for these codes can be found during student work time
- These are all quality codes – you can assign a rating of high even if that aspect of instruction occurs for only a portion of the segment.

Mathematical Practices: Examples (Score for Three Codes)

- Julia: Solving Algebraic Equations
- Karen: Tourist Problem
- Wilhelmina: Polygons and Non-Polygons
- Robert: Factoring

Julia: Solving Algebraic Equations

- 8th grade
- Class is considering two approaches to solving algebraic equations

Julia: Solving Algebraic Equations: Video

How would you score this clip for:

- Multiple Procedures or Solution Methods
- Developing Mathematical Generalizations
- Mathematical Language

- *Take a moment to write down your scores before moving on to our answers…*

Julia: Solving Algebraic Equations: Answers

- Multiple procedures/solutions: **3**
  - There are two different correct solution methods
  - Explicit links between them and a nice discussion of when one method might be preferable to another
- Developing generalizations: **2**
  - Generalization about which method is easier made by student
- Mathematical language: **3**
  - Rich language used by both the teacher and students: distributive property; combine like terms; factor; equation; inverse operation

Karen: Tourist Problem

- 5th grade
- Class has been working on the following problem: “There are 54 tourists in a group. There are twice as many women as men, and there are three times as many children as men. How many women, men, and children are there?”
- At the beginning of the clip, a student shares her solution on the overhead projector

Karen: Tourist Problem: Video

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How would you score this clip for:

- Multiple Procedures or Solution Methods
- Developing Mathematical Generalizations
- Mathematical Language

- *Take a moment to write down your scores before moving on to our answers…*

Karen: Tourist Problem: Answers

- Multiple procedures/solutions: **2**
  - There are two different and correct solutions featured in this clip
  - There is NO explicit, lengthy comparison/discussion (just a brief teacher comment re: one approach is more efficient)
- Developing generalizations: **1**
  - No generalization is being developed
- Mathematical language: **2**
  - Teacher uses mathematical terms (e.g., ratio, quotient), but the usage of mathematical language was not particularly dense

Robert: Factoring

- 9^{th} grade
- Class is working on factorization

Robert: Factoring: Video

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How would you score this clip for:

- Multiple Procedures or Solution Methods
- Developing Mathematical Generalizations
- Mathematical Language

- *Take a moment to write down your scores before moving on to our answers…*

Robert: Factoring: Answers

- Multiple procedures/solutions: **1**
  - Similar problems solved in similar ways
- Developing generalizations: **2**
  - Class develops a correct generalization
  - It is not complete, clear, or detailed
    - In the last 30 seconds, he develops a generalization which is not clear or complete (“The middle one is always half of that.”)
- Mathematical language: **2**
  - Teacher uses several terms (“factoring,” “terms,” “exponent”) meaningfully and correctly, but also some vague words (“it”, “this one”, “doing”, etc.); there are also none of the special features of “high” present

Wilhelmina: Polygons and Non-Polygons

- 6th grade class
- Working on CMP2 *Shapes and Designs*
- The class is introduced to polygons. The teacher projects examples of polygons and non-polygons on the overhead, and asks students to figure out the characteristics of a polygon to complete the definition: "A polygon is a group of line segments put together in a special way."
- We step into the classroom after students have been given some time to identify the characteristics of polygons (by comparing polygons and non-polygons)

Wilhelmina: Polygons and Non-Polygons: Video

How would you score this clip for:

- Multiple Procedures or Solution Methods
- Developing Mathematical Generalizations
- Mathematical Language

- *Take a moment to write down your scores before moving on to our answers…*

Wilhelmina: Polygons and Non-Polygons: Answers

- Multiple procedures/solutions: **1**
  - No multiple solutions featured in this clip
- Developing generalizations: **2**
  - Statements (about characteristics of polygons) generalized from examples on board
  - No part of definition is explicitly incorrect
  - However, the definition of polygon developed here is not complete, clear, or detailed. It omits some properties (a 2D shape; no more than two segments joined at a vertex); there is some ambiguity in how certain characteristics are defined (e.g., “lines always meet,” “it has to have points”)
- Mathematical language: **1**
  - Teacher uses several mathematical terms (e.g., line, line segments, closed figure, etc.) but there is also ambiguity and sloppiness in how these terms are used: “intersect” is used ambiguously; “lines always meet,” “for it to be a closed figure, it has to have points”)

Overall Richness of the Mathematics

- What is the overall richness of the segment?
  - Not an average of the previous codes
  - Ask yourself: Is the segment meaning-oriented and/or full of mathematical practices?

Overall Richness of the Mathematics

- **High (3)**
  - There is a outstanding performance in one or more elements, even for a portion of the segment AND/OR
  - Elements of richness combine to create either a strong sense of mathematical meaning and/or mathematical practices
    - Examples:
      - Focus is on meaning via representations linked to one another or to underlying ideas; explanations that generalize; strong mathematical language
      - Focus is on explicit comparing of solution methods or procedures (e.g., most efficient) but without necessary focus on meaning during this discussion

Overall Richness of the Mathematics

- **Mid (2)**
  - Elements of richness occur, but none are carried off strongly; OR
  - Richness is mixed – for example, some solid elements, but also largely student practice
- **Low (1)**
  - Elements of richness are not present or are minimally present OR
  - Elements of richness are present but incorrect

Overall Richness: Examples

- Lisa: Inverse Operations
- Julia: Solving Algebraic Equations
- Wilhelmina: Polygons and Non-Polygons

Lisa: Inverse Operations

- 5th grade
- Class was working on solving algebraic equations using "inverse operations"

$$n - 2 = 7 \rightarrow n = 2 + 7$$

- However, in some cases this approach did not seem to work

$$31 - n = 12 \not\rightarrow n = 31 + 12$$

$$n \times 2 = 14 \not\rightarrow n = 2 \div 14$$

- The teacher explains why the inverse operation approach can be used in some cases but not in others

Lisa: Inverse Operations: Video

How would you score this clip for:

- Overall Richness

- *Take a moment to write down your score before moving on to our answers…*

Lisa: Inverse Operations: Answers

- Previous richness scores:
  - Linking and connections: **1**
  - Explanations: **1**
- Overall Richness: **1**
  - Most components of richness that are present in the clip (explanations, maybe a developed generalization) are mathematically incorrect

Julia: Solving Algebraic Equations

- 8th grade
- Class is considering two approaches to solving algebraic equations

Julia: Solving Algebraic Equations: Video

How would you score this clip for:

- Overall Richness

- *Take a moment to write down your score before moving on to our answers…*

Julia: Solving Algebraic Equations: Answers

- Previous richness scores
  - Multiple procedures/solutions: **3**
  - Developing generalizations: **2**
  - Mathematical language: **3**
- Overall Richness: **3**
  - On the strength of the truly outstanding performance in the multiple procedures and solution methods code

Wilhelmina: Polygons and Non-Polygons

- 6^{th} grade class
- Working on CMP2 *Shapes and Designs*
- The class is introduced to polygons. The teacher projects examples of polygons and non-polygons on the overhead, and asks students to figure out the characteristics of a polygon to complete the definition: “A polygon is a group of line segments put together in a special way.”
- We step into the classroom after students have been given some time to identify the characteristics of polygons (by comparing polygons and non-polygons)

Wilhelmina: Polygons and Non-Polygons: Video

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How would you score this clip for:

- Overall Richness

- *Take a moment to write down your score before moving on to our answers…*

Wilhelmina: Polygons and Non-Polygons: Answers

- Previous richness scores
  - Multiple procedures/solutions: **1**
  - Developing generalizations: **2**
  - Mathematical language: **1**
- Overall Richness: **2**
  - The generalization activity, while at times incomplete and unclear, dominates the clip and does go a long way towards defining a polygon

Overall Richness: More Examples

- Here are our scores for the overall richness code for the other videos you have watched in the two richness modules:
  - Karen: Long Division: **3**
    - outstanding linking
  - Lauren: Likelihood Line: **3**
    - good linking and explanations
  - Bianca: Integer Subtraction: **1**
    - procedure is confused, no elements of richness

Overall Richness: More Examples

- Karen: Interpreting Remainders: **3**
  - explicit language and meaning-making
- Karen: Tourist Problem: **2**
  - some meaning given, but mostly recitation of procedure
- Robert: Factoring: **1**
  - little richness besides a brief /incomplete generalization

End of Richness Part II

Please move on to Richness Practice module.

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